RESULTS IN SEMI-E-CONVEX FUNCTIONS

Ayache Benhadid

Abbes Laghrour University, Faculty of Science and Technology
Departement of Mathematics and Informatics, Khenchela, Algeria.
e-mail: benhadidayache@yahoo.fr

ABSTRACT

The concept of convexity and its various generalizations is important for quantitative and qualitative studies in operations research or applied mathematics. Recently, E-convex sets and E-convex functions were introduced by Youness [2], and they have some important applications in various branches of mathematical sciences. Youness in [2] introduced a class of sets and functions which is called E-convex sets and E-convex functions by relaxing the definition of convex sets and convex functions. Xiusu Chen [1] introduced a new concept of semi E-convex functions and discussed its properties. According to Xiusu Chen [1], if a function $f : M \to \mathbb{R}$ is semi-E-convex on an E-convex set $M \subset \mathbb{R}^n$ then, $f(E(x)) \leq f(x)$ for each $x \in M$. In this article we have discussed the inverse of this proposition and present some results for convex functions.

1. INTRODUCTION

Youness in [2] introduced a class of sets and functions which is called E-convex sets and E-convex functions by relaxing the definition of convex sets and convex functions. Following this Xiusu Chen [1] introduced a new class of semi-E-convex functions and applied these functions to non linear programming problems see for instance [4, 5] . In this paper, we give weak condition for a lower semi-continuous function on $\mathbb{R}^n$ to be a semi-E-convex function, we also present some results for convex functions.

2. PRELIMINARIES

Let M be a nonempty subset of $\mathbb{R}^n$ and let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a map. We recall :

Definition 1 *[2] A set $M \subseteq \mathbb{R}^n$ is said to be E-convex in $\mathbb{R}^n$ if*

$$tE(x) + (1-t)E(y) \in M,$$

for each $x, y \in M$ and all $t \in [0,1]$.

Definition 2 *[2] A function $f : M \to \mathbb{R}$ is said to be E-convex on M if M is E-convex and*

$$f(tE(x) + (1-t)E(y)) \leq tf(E(x)) + (1-t)f(E(y)),$$

for each $x, y \in M$ and all $t \in [0,1]$.

Definition 3 *[1] A function $f : M \to \mathbb{R}$ is said to be semi-E-convex on M if M is E-convex and*

$$f(tE(x) + (1-t)E(y)) \leq tf(x) + (1-t)f(y),$$

for each $x, y \in M$ and all $t \in [0,1]$.

Definition 4 *[1] We define a map $E \times I$ as follows :*

$$\begin{array}{rcl} E \times I : \mathbb{R}^n \times \mathbb{R} & \rightarrow & \mathbb{R}^n \times \mathbb{R} \\ (x,t) & \rightarrow & (E \times I)(x,t) = (E(x),t). \end{array}$$

This Proposition gives a characterization of a semi-E-convex function in term of its $epi\,(f)$.

Proposition 1 *[1] The function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is semi-E-convex on $\mathbb{R}^n$ if and only if its epigraph $epi\,(f) = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}$ is $E \times I$-convex on $\mathbb{R}^n \times \mathbb{R}$.*

Definition 5 *[3] A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semi-continuous if and only if, for every real number α, the set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is closed.*

In the following we introduce a Proposition about lower semi-continuous functions which shall be used in the sequal. For details and for the missing proofs we refer G. Dal Maso [3].

Proposition 2 *[3] A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semi-continuous if and only if its epigraph is closed.*

Definition 6 *Let $(x,s),(y,t) \in \mathbb{R}^{n+1}$, with $x,y \in \mathbb{R}^n$ and $s,t \in \mathbb{R}$. The line segment $[(x,s),(y,t)]$ (with endpoints (x,s) and (y,t)) is the segment*

$$\{\alpha(x,s) + (1-\alpha)(y,t) : 0 \leq \alpha \leq 1\}.$$

If $(x,s) \neq (y,t)$, the interior $](x,s),(y,t)[$ of $[(x,s),(y,t)]$ is the segment

$$\{\alpha(x,s) + (1-\alpha)(y,t) : 0 < \alpha < 1\}.$$

In a similar way, we can define $[(x,s),(y,t))$ and $((x,s),(y,t)]$.

3. MAIN RESULTS FOR SEMI-E-CONVEX FUNCTIONS

Lemma 3 *Let $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear and idempotent map. Consider $(\bar{x},u) \in [(E(x),s),(E(y),t)]$. Then*

$$E(\bar{x}) = \bar{x}.$$

Proof. Let $(\bar{x},u) \in [(E(x),s),(E(y),t)]$, then there exist $\alpha \in [0,1]$, such that $(\bar{x},u) = \alpha(E(x),s) + (1-\alpha)(E(y),t)$. Using the fact that E is linear and idempotent map, we have

$$\begin{array}{rcl} (E \times I)(\bar{x},u) & = & (E(\alpha E(x) + (1-\alpha)E(y)), \alpha s + (1-\alpha)t) \\ & = & (\alpha E(x) + (1-\alpha)E(y), \alpha s + (1-\alpha)t) \\ & = & (\bar{x},u). \end{array}$$

On the other hand $(E \times I)(\bar{x},u) = (E(\bar{x}),u)$, therfore $E(\bar{x}) = \bar{x}$. ■

We shall make use the following three sets :

$$H_{Sci} = \{f : \mathbb{R}^n \rightarrow \mathbb{R},\ f \text{ is lower semi continuous}\}, \tag{1}$$

$$H_{L,I} = \{E : \mathbb{R}^n \rightarrow \mathbb{R}^n, E \text{ is linear and idempotent}\} \tag{2}$$

and for each $E \in H_{L,I}$ we define H_E as follows :

$$H_E = \{f \in H_{Sci},\ f(E(x)) \leq f(x) \text{ for all } x \in \mathbb{R}^n\} \tag{3}$$

Theorem 4 *Let $E \in H_{L,I}$, and $f \in H_E$. Suppose that there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$, $s, t \in \mathbb{R}$ such that $f(x) < s$, $f(y) < t$,*

$$f(\alpha E(x) + (1-\alpha)E(y)) < \alpha s + (1-\alpha)t.$$

Then f is semi-E-convex.

Proof. By Proposition (1), it is sufficient to show that $epi(f)$ is $E \times I$-convex as a subset of $\mathbb{R}^n \times \mathbb{R}$. By contradiction, suppose that there exist $(x_1, \alpha_1), (x_2, \alpha_2) \in epi(f)$ (with $x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$) and $\alpha_0 \in]0,1[$ such that,
$(\alpha_0 E(x_1) + (1-\alpha_0)E(x_2), \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2) \notin epi(f)$.
Let $x_0 = \alpha_0 E(x_1) + (1-\alpha_0)E(x_2)$ and $\lambda_0 = \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2$, then $(x_0, \lambda_0) \notin epi(f)$. Using the fact that $f \in H_E$, we see that $(E(x_1), \alpha_1), (E(x_2), \alpha_2) \in epi(f)$. Let

$$A = epi(f) \cap [(E(x_1), \alpha_1), (x_0, \lambda_0)]$$

and

$$B = epi(f) \cap [(x_0, \lambda_0), (E(x_2), \alpha_2)].$$

Since $f \in H_{Sci}$, by Proposition (2), $epi(f)$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}$. Consequently, A and B are bounded and closed subsets of $\mathbb{R}^n \times \mathbb{R}$.
Also we have $(x_0, \lambda_0) \notin A$ and $(x_0, \lambda_0) \notin B$. Thus there exist $Z_A = (x_3, \alpha_3) \in A$ and $Z_B = (x_4, \alpha_4) \in B$ such that,

$$\min_{Z \in A} \|Z - (x_0, \lambda_0)\| = \|Z_A - (x_0, \lambda_0)\|$$

and

$$\min_{Z \in B} \|Z - (x_0, \lambda_0)\| = \|Z_B - (x_0, \lambda_0)\|.$$

Hence, we have

$$]Z_A, Z_B[\cap epi(f) = \emptyset. \tag{4}$$

On the other hand, since $Z_A \in epi(f)$ and $Z_B \in epi(f)$, we get
$f(x_3) < \alpha_3 + \varepsilon$, $f(x_4) < \alpha_4 + \varepsilon$ for each $\varepsilon > 0$.
Since $\alpha(\alpha_3 + \varepsilon) + (1-\alpha)(\alpha_4 + \varepsilon) = \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon$. By the hypothesis of the theorem, we obtian

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) < \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon.$$

Since ε is an arbitrary positive real number, it follows that

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) \leq \alpha\alpha_3 + (1-\alpha)\alpha_4. \tag{5}$$

Since $Z_A \in A \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$ and
$Z_B \in B \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$. By Lemma (3) we have $E(x_3) = x_3$ and
$E(x_4) = x_4$. Using (5) we get

$$(\alpha x_3 + (1-\alpha)x_4, \alpha\alpha_3 + (1-\alpha))\alpha_4) \in epi(f).$$

Therfore

$$\alpha Z_A + (1-\alpha)Z_B \in epi(f)$$

which contradicts (4). Thus, we conclude that $epi(f)$ is $E \times I$-convex. ■

Theorem 5 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \alpha f(x) + (1-\alpha)f(y).$$

Proof. Follows from Theorem (4) with $s = f(x) + \varepsilon$ and $t = f(y) + \varepsilon$ for each $\varepsilon > 0$, then taking $\varepsilon \to 0$. ■

Corollary 6 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}\left(E(x) + E(y)\right)\right) \leq \frac{1}{2}\left[f(x) + f(y)\right].$$

Theorem 7 *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \alpha f(x) + (1-\alpha) f(y). \tag{6}$$

If we take $E = Id_{\mathbb{R}^n,}$ we get $E \in H_{L,I}$, and $H_E = H_{Sci}$. Then we find results about convex functions.

Theorem 8 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous.Then f is convex if and only if there exists an $\alpha \in]0,1[$ such that, for all $x, y \in \mathbb{R}^n$,*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Theorem 9 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Corollary 10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}\left[f(x) + f(y)\right].$$

4. REFERENCES

[1] X. Chen, Some properties of semi-E-convex functions, Journal of mathematical Anaysis andApplication, 275, 251-262, (2002).

[2] E. A. Youness, E-convex sets, E-convex functions, and E-convex programming, Journal of Optimization Theory and Applications, 102, 439-450, (1999).

[3] G. Dal maso, An Introduction to Γ-convergence,10-11. BirKhauser, Boston, (1993).

[4] A. A. Megahed, H. G. Gomma, E. A. Youness, A. H. Banna, Optimality conditions of *E*-convex programing for an *E*-differentiable function, journal of Inequalities and Applications, 246, 1-11, (2013).

[5] M. A. Noor, M. U. Awan, K. I. Noor, On some inequalities for relative semi-convex functions, journal of Inequalities and Applications, 332, 1-16,(2013).