A NEW APPROXIMATE ANALYTICAL SOLUTION OF FRACTIONAL ORDER NONLINEAR WAVE-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT

In this work, we propose a new modified technique known as fractional Aboodh projected differential transform method (FAPDTM) to solve fractional order nonlinear wave-like equations with variable coefficients. The fractional derivative is described in the sense of Caputo. This method is the combination of two powerful methods : the Aboodh transform method and projected differential transform method. We obtain the solutions in the form of series which is rapidly converges to the exact solution. Three numerical examples are presented to illustrate the validity and applicability of our proposed technique.

1. INTRODUCTION

Recently, nonlinear fractional partial differential equations (NFPDEs) were tackled by many researchers because they play an important role in describing many phenomena arising in physics, chemistry, biology, aerodynamics, control theory, finance, and social sciences [2, 8, 13, 14].

The exact solutions of the NFPDEs can help us to know the described process. So, in the past decades, mathematicians have made many efforts in the study of exact solutions of NFPDEs. But, for most these equations, no exact solution is known and, in some cases, it is not even clear whether a unique solution exists. So, approximation methods, such as numerical and analytical methods, have been developed.

Several numerical and analytical methods have been proposed for the solutions of NFPDEs such as : Adomian decomposition method (ADM) [15], homotopy analysis method (HAM) [17], homotopy perturbation method (HPM) [6], generalized differential transform method (GDTM) [3], fractional variational iteration method (FVIM) [16], fractional residual power series method (FRPSM) [9], generalized Taylor fractional series method (GTFSM) [10].

The main objective of this work, is to determine a new approximate analytical solution of fractional order nonlinear wave-like equations with variable coefficients of the form

$$\begin{aligned} D_t^{\alpha} u &= \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i},u_{x_j}) \\ &\quad + \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X,t,u) + S(X,t), \qquad (1) \end{aligned}$$

subject to the initial conditions

$$u(X,0) = a_0(X),\ u_t(X,0) = a_1(X), \qquad (2)$$

where D_t^{α} is the Caputo fractional derivative operator of order α with $1 < \alpha \leq 2$.

Here, $u = \{u(X,t), X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, t \geq 0\}$ is an unknown function, F_{1ij}, G_{1i} $i, j \in \{1, 2, ..., n\}$ are nonlinear functions of X, t and u, F_{2ij}, G_{2i} $i, j \in \{1, 2, ..., n\}$, are nonlinear functions of derivatives of u with respect to x_i and x_j $i, j \in \{1, 2, ..., n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

The outline of the paper is as follows. In Section 2, we present some fundamental definitions of the fractional calculus and the Aboodh transform. In Section 3, we introduce our results of the fractional Aboodh projected differential transformation method (FAPDTM) for fractional order nonlinear wave-like equations with variable coefficients (1) with the initial conditions (2). In Section 4, we propose three numerical examples in order to show the validity and effectiveness of this method. Moreover, we present our obtained results (Graphs and Tables) comparing them with exact solutions. These results were verified using MATLAB Software. Finally, in Section 5, we give a conclusion of this work.

2. DEFINITIONS AND PRELIMINARIES

In this section, we give some definitions and important properties of the fractional calculus theory and the Aboodh transform which shall be used in this paper.

Definition 1 *[12] A real function $f(t), t > 0$, is considered to be in the space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h(t) \in C([0, \infty[)$, and it is said to be in the space C_{μ}^n if $f^{(n)} \in C_{\mu}$, $n \in \mathbb{N}$.*

Definition 2 *[12] The Riemann-Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ for a function $f \in C_{\mu}, \mu \geq -1$ is defined as follows*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \ t > 0. \tag{3}$$

Definition 3 *[12] The Caputo fractional derivative operator of order $n-1 < \alpha \leq n$ for a function $f \in C_{-1}^n$ is de ned as follows*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi. \tag{4}$$

Definition 4 *[12] The Mittag-Leffler function is defined as follows*

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{5}$$

A further generalization of (5) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{6}$$

For $\alpha = 1$, $E_{\alpha}(z)$ reduces to e^z.

Definition 5 *[1] The Aboodh transform is defined over the set of functions*

$$A = \left\{ f(t) \mid \exists M, k_1, k_2 > 0, |f(t)| < Me^{k_j|t|}, \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following integral

$$\mathscr{A}[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, t \geq 0, k_1 < v < k_2, \tag{7}$$

where v is the factor of the variable t.

Some basic properties of the Aboodh transform are given as follows :

Property 1 : The Aboodh transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathscr{A}[\lambda f(t) \pm \mu g(t)] = \lambda \mathscr{A}[f(t)] \pm \mu \mathscr{A}[g(t)].$$

Property 2 : If $f^{(n)}(t)$ is the $n-$th derivative of the function $u(t) \in A$ with respect to "t" then its Aboodh transform is given by

$$\mathscr{A}\left[f^{(n)}(t)\right] = v^n K(v) - \sum_{k=0}^{n-1} v^{n-2-k} f^{(k)}(0).$$

Property 3 : Some special Aboodh transforms

$$\begin{aligned}
\mathscr{A}(1) &= \frac{1}{v^2}, \\
\mathscr{A}(t) &= \frac{1}{v^3}, \\
\mathscr{A}\left[\frac{t^n}{n!}\right] &= \frac{1}{v^{n+2}}, n = 0, 1, 2, ... \\
\mathscr{A}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \frac{1}{v^{\alpha+2}}, \alpha \geq 0.
\end{aligned}$$

Theorem 1 *[11] Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $K(v)$ be the Aboodh transform of the function $f(t)$, then the Aboodh transform of the Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathscr{A}[D^\alpha f(t)] = v^\alpha K(v) - \sum_{k=0}^{n-1} v^{\alpha-2-k} f^{(k)}(0). \tag{8}$$

3. FAPDTM FOR FRACTIONAL ORDER NONLINEAR WAVE-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS

Theorem 2 *Consider the following fractional order nonlinear wave-like equations with variable coefficient(1) subject to the initial conditions (2). Then, by FAPDTM the approximate analytical solution of Eqs. (1) and (2) is given in the form of infinite series as follows*

$$u(X,t) = \sum_{k=0}^{\infty} U(X,k),$$

where $U(X,k)$ is the projected differential transformed function.

Proof. In order to to achieve our goal, we consider the following fractional order nonlinear wave-like equations with variable coefficients (1) subject to the initial conditions (2).

Applying the Aboodh transform on both sides of (1) subject to initial conditions (2) and using the theorem 1, we get

$$\begin{aligned}\mathscr{A}[u(X,t)] &= \frac{1}{v^{\alpha}}\sum_{k=0}^{n-1} v^{\alpha-2-k}u^{(k)}(X,0)+\frac{1}{v^{\alpha}}\mathscr{A}[S(X,t)] \\ &+\frac{1}{v^{\alpha}}\mathscr{A}\Bigg[\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{x_i},u_{x_j}) \\ &+\sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{x_i})+H(X,t,u)\Bigg]. \qquad (9)\end{aligned}$$

Taking the inverse Aboodh transform on both sides of (9), we have

$$\begin{aligned}u(X,t) &= L(X,t)+\mathscr{A}^{-1}\Bigg(\frac{1}{v^{\alpha}}\mathscr{A}\Bigg[\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{x_i},u_{x_j}) \\ &+\sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{x_i})+H(X,t,u)\Bigg]\Bigg), \qquad (10)\end{aligned}$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions.

Now, we apply the projected differential transform method [7] to Eq. (10), we get

$$\begin{aligned}U(X,0) &= L(X,t), \\ U(X,k+1) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}[A(X,k)+B(X,k)+C(X,k)]\right), k\geq 0, \qquad (11)\end{aligned}$$

where $A(X,k), B(X,k)$ and $C(X,k)$ are transformed form of the nonlinear terms, $\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{x_i},u_{x_j})$, $\sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{x_i})$ and $H(X,t,v)$, rerspectively.

From Eq. (11) we have

$$\begin{aligned}U(X,0) &= L(X,t), \\ U(X,1) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}[A(X,0)+B(X,0)+C(X,0)]\right), \\ U(X,2) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}[A(X,1)+B(X,1)+C(X,1)]\right), \\ U(X,3) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}[A(X,2)+B(X,2)+C(X,2)]\right), \\ &\vdots\end{aligned}$$

and so on.

Then, the approximate analytical of Eqs. (1) and (2) is given as follows

$$u(X,t)=\sum_{k=0}^{\infty} U(X,k).$$

The proof is complete. ■

Remark 1 *The $n-$term approximate solution of Eqs. (1) and (2) is given by*

$$u(X,t)=\sum_{k=0}^{n-1}U(X,k)=U(X,0)+U(X,1)+U(X,2)+...+U(X,n-1). \tag{12}$$

4. NUMERICAL EXAMPLES

In this section, we apply the FAPDTM on three examples of nonlinear wave-like equations with Caputo time-fractional derivative and then compare our approximate solutions with the exact solutions.

Example 1 *Consider the following two dimensional fractional order nonlinear wave-like equations with variable coefficients*

$$D_t^{\alpha}u=\frac{\partial^2}{\partial x\partial y}(u_{xx}u_{yy})-\frac{\partial^2}{\partial x\partial y}(xyu_xu_y)-u, 1<\alpha\leq 2, \tag{13}$$

subject to the initial conditions

$$u(x,y,0)=e^{xy},\ u_t(x,y,0)=e^{xy}, \tag{14}$$

where $u=\{u(x,y,t),(x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+\}$.

By applying the steps involved in FAPDTM as presented in Section 3 to Eqs. (13) and (14), we have the following iteration formula

$$\begin{aligned} U(x,y,0) &= e^{xy}+te^{xy}, \\ U(x,y,k+1) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}\left[\frac{\partial^2}{\partial x\partial y}A(x,y,k)-\frac{\partial^2}{\partial x\partial y}B(x,y,k)-U(x,y,k)\right]\right), \end{aligned} \tag{15}$$

where $A(x,y,k)$ and $B(x,y,k)$ are transformed form of the nonlinear terms, $u_{xx}u_{yy}$ and xyu_xu_y.

For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A(0) &= U_{xx}(0)U_{yy}(0), \\ A(1) &= U_{xx}(0)U_{yy}(1)+U_{xx}(1)U_{yy}(0), \\ A(2) &= U_{xx}(0)U_{yy}(2)+U_{xx}(1)U_{yy}(1)+U_{xx}(2)U_{yy}(0), \end{aligned}$$

$$\begin{aligned} B(0) &= xyU_x(0)U_y(0), \\ B(1) &= xyU_x(0)U_y(1)+xyU_x(1)U_y(0), \\ B(2) &= xyU_x(0)U_y(2)+xyU_x(1)U_y(1)+xyU_x(2)U_y(0). \end{aligned}$$

From the relationship in (15), we obtain

$$\begin{aligned} U(x,y,0) &= (1+t)e^{xy}, \\ U(x,y,1) &= -\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\ U(x,y,2) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}, \\ &\vdots \end{aligned}$$

and so on.

Then, the approximate analytical solution of Eqs. (13) and (14) can be expressed by

$$\begin{aligned} u(x,y,t) &= \left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy} \\ &= \left(E_{\alpha}(-t^{\alpha})+tE_{\alpha,2}(-t^{\alpha})\right)e^{xy}, \end{aligned}$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ are the Mittag-Leffler functions, defined by Eqs. (5) and (6).

Taking $\alpha=2$, the approximate analytical solution of Eqs. (13) and (14) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x,y,t)=\left(1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}-...\right)e^{xy}.$$

So, the exact solution of Eqs. (13) and (14) in a closed formof elementary function will be

$$u(x,y,t)=(\cos t+\sin t)\,e^{xy}.$$

The above two expressions is exactly same as those given by ADM [4] and HPTM [5].

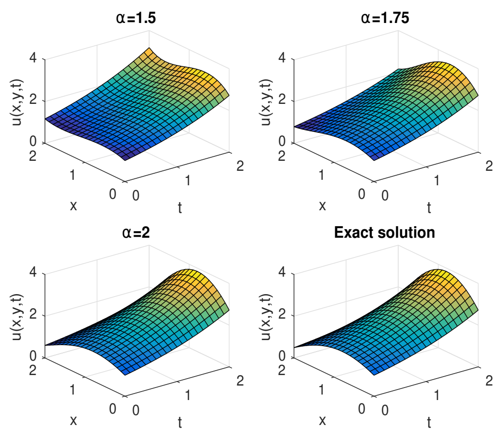

FIGURE 1 – The surface graph of the 3−term approximate solutions by FAPDTM and exact solution for Example 1 when $y=0.5$.

$t/x,y$	0.1	0.3	0.5	0.7
0.1	1.4226×10^{-9}	1.5411×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}
0.3	1.0648×10^{-6}	1.1535×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}
0.5	2.3382×10^{-5}	2.5330×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}
0.7	1.8000×10^{-4}	1.9499×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}
0.9	8.2963×10^{-4}	8.9872×10^{-4}	1.0547×10^{-3}	1.3407×10^{-3}

TABLE 1 – Comparison of the absolute errors for the 3−term approximate solutions by FAPDTM and exact solution for Example 1, when $\alpha=2$.

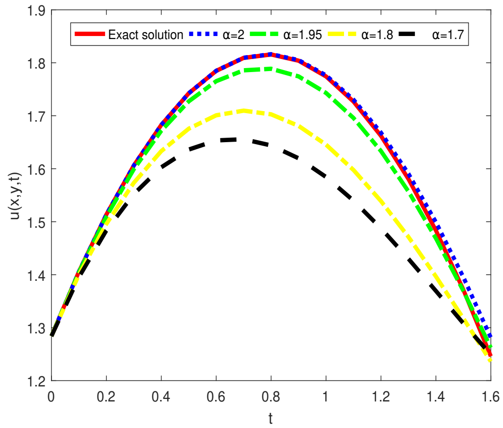

FIGURE 2 – The behavior of the 3−term approximate solutions by FAPDTM and exact solution for Example 1 when $x = y = 0.5$.

Example 2 *Consider the following one dimensional fractional order nonlinear wave-like equations with variable coefficients*

$$D_t^{\alpha} u = u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u, \ 1 < \alpha \leq 2, \tag{16}$$

subject to the initial conditions

$$u(x,0) = e^x, u_t(x,0) = e^x, \tag{17}$$

where $u = \{u(x,t), (x,t) \in]0,1[\times \mathbb{R}^+\}$.

By applying the steps involved in FAPDTM as presented in Section 3 to Eqs. (16) and (17), we have the following iteration formula

$$\begin{aligned} U(x,0) &= (1+t)e^x, \\ U(x,k+1) &= \mathscr{A}^{-1}\left(\frac{1}{v^{\alpha}}\mathscr{A}\left[A(x,k) + B(x,k) - 18C(x,k) + U(x,k)\right]\right), \end{aligned} \tag{18}$$

where $A(x,k)$,$B(x,k)$ *and* $C(x,k)$ *are transformed form of the nonlinear terms,* $u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})$, $u_x^2 \frac{\partial^2}{\partial x^2}(u_{xx}^3)$, *and* u^5, *rerspectively.*

For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A(0) &= U^2(0)\frac{\partial^2}{\partial x^2}\left[U_x(0)U_{xx}(0)U_{xxx}(0)\right], \\ A(1) &= 2U(0)U(1)\frac{\partial^2}{\partial x^2}\left[U_x(0)U_{xx}(0)U_{xxx}(0)\right] + U^2(0)\frac{\partial^2}{\partial x^2}\left[U_x(1)U_{xx}(0)U_{xxx}(0)\right. \\ &\quad \left. + U_x(0)U_{xx}(1)U_{xxx}(0) + U_x(0)U_{xx}(0)U_{xxx}(1)\right], \end{aligned}$$

$$\begin{aligned} B(0) &= U_x^2(0)\frac{\partial^2}{\partial x^2}U_{xx}^3(0), \\ B(1) &= 2U_x(0)U_x(1)\frac{\partial^2}{\partial x^2}U_{xx}^3(0)+3U_x^2(0)\frac{\partial^2}{\partial x^2}\left[U_{xx}^2(0)U_{xx}(1)\right], \end{aligned}$$

$$\begin{aligned} C(0) &= U^5(0), \\ C(1) &= 5U^4(0)U(1). \end{aligned}$$

From the relationship in (18), we obtain

$$\begin{aligned} V(x,0) &= (1+t)e^x, \\ V(x,1) &= \left(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x, \\ V(x,2) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^x, \\ &\vdots \end{aligned}$$

and so on

Then, the approximate analytical solution of Eqs. (16) and (17) can be expressed by

$$\begin{aligned} u(x,t) &= \left(1+t+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+...\right)e^x \\ &= \left(E_\alpha(t^\alpha)+tE_{\alpha,2}(t^\alpha)\right)e^x, \end{aligned}$$

where $E_\alpha(t^\alpha)$ and $E_{\alpha,2}(t^\alpha)$ are the Mittag-Leffler functions, defined by Eqs. (5) and (6).

Taking $\alpha = 2$, the approximate analytical solution of Eqs. (16) and (17) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x,t) = \left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+...\right)e^x.$$

So, the exact solution of Eqs. (16) and (17) in a closed form of elementary function will be

$$u(x,t) = e^{x+t}.$$

The above two expressions is exactly same as those given by ADM [4] and HPTM [5].

t/x	0.1	0.3	0.5	0.7
0.1	1.5572×10^{-9}	1.9019×10^{-9}	2.3230×10^{-9}	2.8373×10^{-9}
0.3	1.1688×10^{-6}	1.4276×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}
0.5	2.5810×10^{-5}	3.1525×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}
0.7	2.0036×10^{-4}	2.4472×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}
0.9	9.3372×10^{-4}	1.1404×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}

TABLE 2 – Comparison of the absolute errors for the 3−term approximate solutions by FAPDTM and exact solution for Example 2, when $\alpha = 2$.

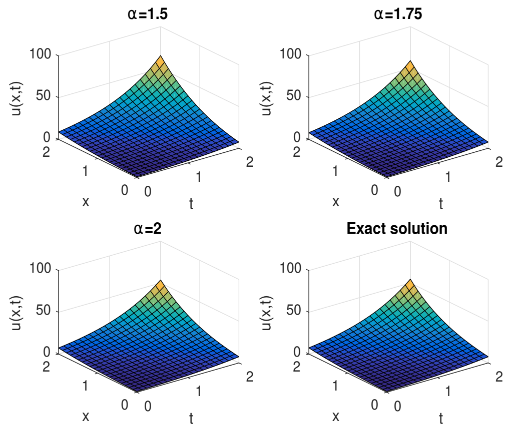

FIGURE 3 – The surface graph of the $3-$term approximate solutions by FAPDTM and exact solution for Example 2.

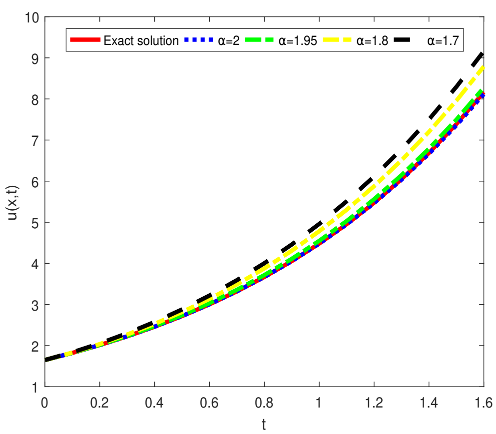

FIGURE 4 – The behavior of the $3-$term approximate solutions by FAPDTM and exact solution for Example 2 when $x = 0.5$.

Example 3 *Consider the following one dimensional fractional order nonlinear wave-like equations with variable coefficients*

$$D_t^{\alpha} u = x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx}^2) - u, 1 < \alpha \leq 2, \tag{19}$$

subject to the initial conditions

$$u(x,0) = 0, u_t(x,0) = x^2, \tag{20}$$

where $u = \left\{u(x,t),(x,t) \in]0,1[\times \mathbb{R}^+\right\}$.

By applying the steps involved in FAPDTM as presented in Section 3 to Eqs. (19) and (20), we have the following iteration formula

$$\begin{aligned} U(x,0) &= tx^2, \\ U(x,k+1) &= \mathscr{A}^{-1}\left(\frac{1}{v^\alpha}\mathscr{A}\left[x^2\frac{\partial}{\partial x}A(x,k) - x^2B(x,k) - U(x,k)\right]\right), \end{aligned} \tag{21}$$

where $A(x,k)$ *and* $B(x,k)$ *are transformed form of the nonlinear terms,* $u_x u_{xx}$ *and* u_{xx}^2 *.*

For the convenience of the reader, the first few nonlinear terms are asfollows

$$\begin{aligned} A(0) &= U_x(0)U_{xx}(0), \\ A(1) &= U_x(0)U_{xx}(1) + U_x(1)U_{xx}(0), \\ A(2) &= U_x(0)U_{xx}(2) + U_x(1)U_{xx}(1) + U_x(2)U_{xx}(0), \end{aligned}$$

$$\begin{aligned} B(0) &= U_{xx}^2(0), \\ B(1) &= 2U_{xx}(0)U_{xx}(1), \\ B(2) &= 2U_{xx}(0)U_{xx}(2) + U_{xx}^2(1). \end{aligned}$$

From the relationship in (21), we obtain

$$\begin{aligned} U(x,0) &= tx^2, \\ U(x,1) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ U(x,2) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

and so on.

Then, the approximate analytical solution of Eqs. (19) and (20) can be expressed by

$$\begin{aligned} u(x,t) &= x^2\left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - ...\right) \\ &= x^2\left(tE_{\alpha,2}(-t^\alpha)\right), \end{aligned}$$

where $E_{\alpha,2}(-t^\alpha)$ *is the Mittag-Leffler function, defined by Eq. (5).*

Taking $\alpha = 2$*, the approximate analytical solution of Eqs. (19) and (20) has the general pattern form which is coinciding with the following exact solution in terms of infinite series*

$$u(x,t) = x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - ...\right).$$

So, the exact solution of Eqs. (19) and (20) in a closed form of elementary function will be

$$u(x,t) = x^2 \sin t.$$

The above two expressions is exactly same as those given by ADM [4] and HPTM [5].

Remark 2 *In this paper, we only apply three terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.*

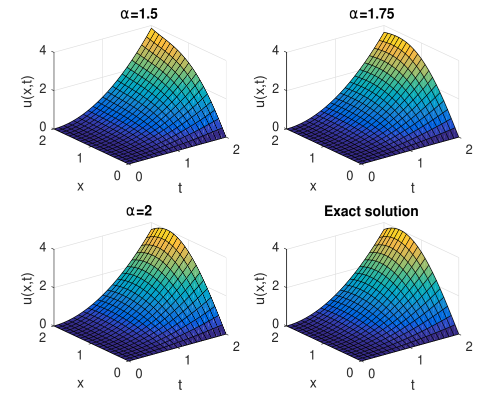

FIGURE 5 – The surface graph of the 3−term approximate solutions by FAPDTM and exact solution for Example 3.

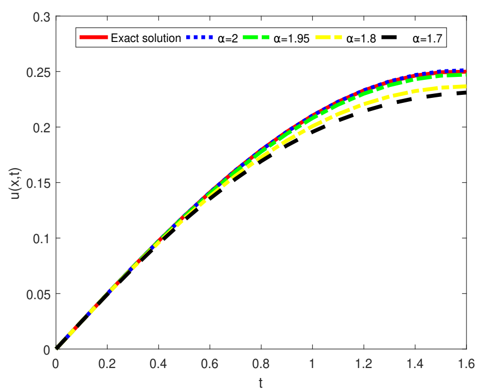

FIGURE 6 – The behavior of the 3−term approximate solutions by FAPDTM and exact solution for Example 3 when $x = 0.5$.

t/x	0.1	0.3	0.5	0.7
0.1	1.9839×10^{-13}	1.7855×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}
0.3	4.3339×10^{-10}	3.9005×10^{-9}	1.0835×10^{-8}	2.1236×10^{-8}
0.5	1.5447×10^{-8}	1.3903×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}
0.7	1.6229×10^{-7}	1.4606×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}
0.9	9.3840×10^{-7}	8.4456×10^{-6}	2.3460×10^{-5}	4.5982×10^{-5}

TABLE 3 – Comparison of the absolute errors for the 3−term approximate solutions by FAPDTM and exact solution for Example 3 when $\alpha = 2$.

5. CONCLUSIONS

In this work, the fractional Aboodh projected differential transform method (FAPDTM) has been successfully applied to study the nonlinear wave-like equations with Caputo time-fractional derivative .The results show that the FAPDTM is an efficient and easy to use technique for finding approximate analytical solution for this equation. The obtained approximate solution using the suggested method is in excellent agreement with the exact solution. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of nonlinear fractional partial differential equations.

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