EXPLICIT LIMIT CYCLE FOR CLASS OF MULTI-PARAMETER POLYNOMIAL DIFFERENTIAL SYSTEM

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ABSTRACT

In this paper, we establish several necessary conditions to confirm the existence of a non-algebraic limit cycle for class of multi-parameter polynomial differential system, Moreover this limit cycle is explicitly given in polar coordinates. Concrete example exhibiting the applicability of our result are introduced.

1. INTRODUCTION

We consider here two-dimensional polynomial differential systems of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y), \\ \dot{y} = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

where P and Q are two polynomials of $\mathbb{R}[x,y]$ denotes the ring of polynomials in the variables x and y with real coefficients. By definition, the degree of the system (1) is $n=\max(\deg(P),\deg(Q))$. A limit cycle of system (1) is an isolated periodic orbit and it is said to be algebraic if it is contained in the zero set of an algebraic curve, otherwise it is called non-algebraic.

Three of the main problems appear in the qualitative theory of real planar differential systems which are the determination of centres, limit cycles, and first integrals of a system of the form (1).

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant C^1 function $H:\Omega\to\mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH}{dt} = P(x,y)\frac{\partial H(x,y)}{\partial x} + Q(x,y)\frac{\partial H(x,y)}{\partial y} = 0, \text{ in the points of } \Omega.$$

Moreover, $H=h$ is the general solution of this equation, where h is an arbitrary constant.

We recall that an algebraic curve defined by $u(x,y)=0$ is an invariant curve for (1) if there exists a polynomial $k(x,y)$ (called the cofactor) such that

$$P(x,y)\frac{\partial u(x,y)}{\partial x}+Q(x,y)\frac{\partial u(x,y)}{\partial y}=K(x,y)\,u(x,y)\,.$$

It is very difficult to detect if a planar differential system is integrable or not and also to know if the limit cycles for this system exist and are algebraic, as well as the determination of their explicit expressions.

In the beginning, the explicit expressions of limit cycles were algebraic (see, for example, [[3], [6], [10]] and references therein. On the other hand, it seems intuitively clear that "most" limit cycles of planar polynomial vector fields have to be non algebraic. Nevertheless, until 1995 it was not proved that the limit cycle of the van der Pol equation is not algebraic, see K. Odani [11]. The van der Pol system can be written as a polynomial differential system (1) of degree 3, but its limit cycle is not known explicitly. Until recently, the only limit cycles known in an explicit way were algebraic. In the chronological order the first examples were explicit non-algebraic limit cycles appeared are those of A. Gasull and all [8] and by Al-Dosary, Khalil I. T.[2] for $n=5$. In [5], an example of an explicit limit cycle which is not algebraic is given for $n=3$.

In this paper, we introduce a multi-parameter polynomial differential system

$$\begin{cases} \dot{x}=\left(x^2+y^2\right)\left(x-4y^5-4x^2y^3\right)+x\left(x^4+y^4-\delta\right)\left(\alpha x^2-4\beta xy+\alpha y^2\right) \\ \dot{y}=\left(x^2+y^2\right)\left(y+4x^5+4x^3y^2\right)+y\left(x^4+y^4-\delta\right)\left(\alpha x^2-4\beta xy+\alpha y^2\right)\end{cases}. \quad (2)$$

where α,β,δ are real constants. We prove the existence of a non-algebraic limit cycle. Moreover this limit cycle is explicitly given in polar coordinates. Concrete example exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$\begin{aligned} N(\zeta) &= \frac{(8\beta h\sin 2\zeta-4\alpha h+4)}{\cos 4\zeta+3} \\ M(\zeta) &= \frac{\alpha\cos 4\zeta-5\beta\sin 2\zeta-\beta\sin 6\zeta+4\sin 4\zeta+3\alpha}{\cos 4\zeta+3} \end{aligned}$$

2. MAIN RESULT

We prove the following result.

Theorem 1 *Consider a multi-parameter polynomial differential system (2)*
Then the following statements hold
1) The system (2) is Darboux integrable with the first integral in the form

$$F(x,y)=\frac{\left(x^2+y^2\right)^2}{\exp\left(\int_0^{\arctan\frac{y}{x}}M(\zeta)d\zeta\right)}-\left(\int_0^{\arctan\frac{y}{x}}\exp\left(-\int_0^{\zeta}M(\omega)d\omega\right)N(\zeta)d\zeta\right).$$

2) If $6|\beta|+|\alpha|++3\alpha+4<0$ *and* $4\alpha\delta+8|\beta\delta|-4<0$*, then the he system (2) has non-algebraic limit cycle explicitly given in polar coordinates* (r,θ) *by :*

$$r(\theta,r^*)=\left(\exp\left(\int_0^{\theta}M(\zeta)d\zeta\right)\left(\sqrt[4]{\int_0^{\theta}N(\zeta)\exp\left(-\int_0^{\zeta}M(\omega)d\omega\right)d\zeta+(r^*)^4}\right)\right),$$

where

$$r^* = \left(\sqrt[4]{\frac{\int_0^{2\pi} -N(\zeta)\exp\left(-\int_0^{\zeta} M(\omega)d\omega\right)d\zeta}{1-\exp\left(-\int_0^{2\pi} M(\zeta)d\zeta\right)}} \right).$$

Moreover, this limit cycle is a stable hyperbolic limit cycle.

Example 1 *If we take $\alpha=-4, \beta=0$ and $\delta=1$, then system (2) reads*

$$\begin{cases} \dot{x} = x\left(-1+x^4+y^4\right)\left(-4x^2-4y^2\right)-\left(x^2+y^2\right)\left(-x+4x^2y^3+4y^5\right), \\ \dot{y} = y\left(-1+x^4+y^4\right)\left(-4x^2-4y^2\right)+\left(x^2+y^2\right)\left(y+4x^3y^2+4x^5\right), \end{cases} \tag{3}$$

this system has a non-algebraic limit cycle whose expression in polar coordinates is (r,θ)

$$\begin{aligned} r(\theta,r^*) &= \left((r^*)^4+\int_0^{\theta}\frac{20}{\cos 4\zeta+3}\exp\left(-\int_0^{\zeta}\frac{-12+4\sin 4w-4\cos 4w}{\cos 4w+3}dw\right)d\zeta\right)^{\frac{1}{4}} \\ &\quad \times \exp\left(\frac{1}{4}\int_0^{\theta}\frac{-12+4\sin 4\zeta-4\cos 4\zeta}{\cos 4\zeta+3}d\zeta\right), \end{aligned}$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is

$$r^* = \sqrt[4]{\frac{\int_0^{2\pi} -\frac{20}{\cos 4\zeta+3}\exp\left(-\int_0^{\zeta}\frac{-12+4\sin 4w-4\cos 4w}{\cos 4w+3}dw\right)d\zeta}{1-\exp\left(-\int_0^{2\pi}\frac{-12+4\sin 4\zeta-4\cos 4\zeta}{\cos 4\zeta+3}d\zeta\right)}}.$$

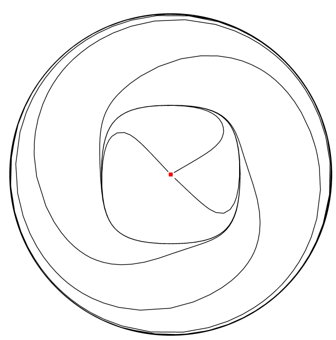

FIGURE 1 – The phase portrait in the Poincaré disc of the system (3)

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