EXPONENTIAL STABILITY FOR A DELAYED FLEXIBLE STRUCTURE

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ABSTRACT

In this paper, we consider a non-uniform and delayed flexible structure with temperature and micro-temperature effects. We prove the well-posed of the problem using semi-groups theory, as well as an exponential stability using the multiplier method under a small condition on time delay.

1. INTRODUCTION

In this paper, we aim to study the following inhomogeneous flexible structure system with micro-temperature effect :

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+dw_x+\eta\theta_x+\mu u_t\left(x,t-\tau_0\right)=0, \\ c\theta_t-k\theta_{xx}+\eta u_{tx}+k_1w_x=0, \\ \tau w_t-k_3w_{xx}+k_2w+k_1\theta_x+du_{tx}=0, \end{cases} \tag{1}$$

where $u(x,t)$ is the displacement of a particle at position $x\in(0,L)$ and time $t>0$, θ and w are the temperature of the body and the micro-temperature vector respectively. $\eta>0$ is the coupling constant, that accounts for the heating effect, and $k,k_1,k_2,k_3,c,d,\tau>0$. $m(x)$, $\delta(x)$ and $p(x)$ are responsible for the non-uniform structure of the body, and, respectively, denote mass per unit length of structure, coefficient of internal material damping and a positive function related to the stress acting on the body at a point x. We consider the following initial and boundary conditions :

$$\begin{array}{l} u\left(.,0\right)=u_0\left(x\right),\ u_t\left(.,0\right)=u_1\left(x\right),\ \theta\left(.,0\right)=\theta_0\left(x\right),w\left(.,0\right)=w_0\left(x\right),\ \ \forall x\in[0,L] \\ u\left(0,t\right)=u\left(L,t\right)=\theta\left(0,t\right)=\theta\left(L,t\right)=w_x\left(0,t\right)=w_x\left(L,t\right)=0,\ \forall t\geq0. \end{array} \tag{2}$$

In the presence of second sound, Alves et al. [2] concerned with the system

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\eta\theta_x=0, \\ \theta_t+kq_x+\eta u_{tx}=0, \\ \tau q_t+\beta q+k\theta_x=0, \end{cases} \tag{3}$$

They established the well-posedness of the system and proved its stability exponential and polynomial under suitable boundary conditions. We know that the dynamic systems with delay terms have become a major research subject in differential equation since the 1970s of the last century. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, non-instant transmission phenomena and biological motivations. Moreover, Datko et al. [8] in 1986 showed that the presence of the delay may not only destabilize a system which is asymptotically stable in the absence of the delay but may also lead to ill-posedness (see also [22] and [23]). Li et al. [18] considered (3) with a delay term of the form $\mu u_t(x,t-\tau_0)$ in its first equation,

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\eta\theta_x+\mu u_t(x,t-\tau_0)=0, \\ \theta_t+kq_x+\eta u_{tx}=0, \\ \tau q_t+\beta q+k\theta_x=0, \end{cases} \tag{4}$$

they proved that the system is exponential decay under a "small" condition on time delay. For more details discussion on the subject see [1, 10] and the references therein.

The linear theory of thermo-elasticity with micro-temperatures for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess micro-temperatures was constructed by Ieşan and Quintanilla [15, 17]. The work is motivated by increasing use of materials which possess thermal variation at a microstructure level. The same authors proved an existence theorem and established the continuous dependence of solutions of the initial data and body loads. Originally, Grot [11] was the first to take into consideration the inner structure of a body in order to develop a thermodynamic theory for thermo-elastic materials where micro-elements, in addition to classic micro-deformations, possess micro-temperatures. While, the fundamental solution of the equations of the theory of thermo-elasticity with micro-temperatures was constructed by Svanadze [28]. Riha [24, 25] developed a further study concerning heat conduction in thermo-elastic materials with inner structure. We note that the concept of micro-temperature was just used in the theory of thermodynamics for elastic materials with microstructure. In addition to micro-deformations of the string, the micro-elements of the continuum possess micro-temperatures which represent the variation of the temperature within a micro-volume. We refer the interested readers to [3, 5, 6, 7, 9, 12, 13, 14, 16, 19, 26, 27] for details discussion on the theory.

Motivated by works mentioned above, we investigate (1)-(2) under suitable condition and establish the well-posedness of the problem using semi-group theory, as well as the stability result of the solution using the multiplier method. Our purpose in the present manuscript is to obtain an exponential decay rate estimates of the energy function of (1) with a "small" condition on time delay. Introducing the microtemperature damping instead of the second sound together with the delay term in the internal feedback of flexible structure makes our problem different from those considered so far in the literature.

This paper is organized as follows ; In the second section, we introduce some assumptions needed in our work then prove the well-posedness of the system (1)-(2). In the last section we state and prove our stability result.

2. WELL-POSEDNESS OF THE PROBLEM

In this section, we present some assumptions and give the existence and uniqueness result of system (1)-(2) using the semigroup theory. Throughout this paper, c' represents a generic positive constant and is different in various occurrences.

Taking the following new variable

$$z(x,\rho,t)=u_t(x,t-\rho\tau_0),\ \text{in}\ (0,L)\times(0,1)\times(0,\infty),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,t)+z_\rho(x,\rho,t)=0, \\ z(x,0,t)=u_t(x,t). \end{cases}$$

Consequently, problem (1)-(2) is equivalent to

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+dw_x+\eta\theta_x+\mu z(x,1,t)=0, \\ c\theta_t-k\theta_{xx}+\eta u_{tx}+k_1w_x=0, \\ \tau w_t-k_3w_{xx}+k_2w+k_1\theta_x+du_{tx}=0, \\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, \\ u(.,0)=u_0(x),\ u_t(.,0)=u_1(x),\ \theta(.,0)=\theta_0(x), w(.,0)=w_0(x), \\ u(0,t)=u(L,t)=\theta(0,t)=\theta(L,t)=w_x(0,t)=w_x(L,t)=0,\ \forall t\geq 0, \\ z(x,0,t)=u_t(x,t) \\ z(x,\rho,0)=f_0(x,-\rho\tau_0) \end{cases} \tag{5}$$

where f_0 is the history function.

The aim of this section is to prove that system (1) is well-posed. From Equation $(1)_3$ and the boundary conditions, we have

$$\frac{d}{dt}\int_0^L w(x,t)\,dx+\frac{k_2}{\tau}\int_0^L w(x,t)\,dx=0,\ \forall t\geq 0,$$

thus

$$\int_0^L w(x,t)\,dx=\left(\int_0^L w_0dx\right)\exp\left(\frac{-t}{\tau}k_2\right),\ \forall t\geq 0,$$

So, if we set

$$\widetilde{w}(x,t)=w(x,t)-\frac{1}{L}\left(\int_0^L w_0dx\right)\exp\left(\frac{-t}{\tau}k_2\right),$$
$$t\geq 0,\ x\in[0,L],$$

then, $(u,u_t,\theta,\widetilde{w},z)$ satisfies Equation (1), and

$$\int_0^L \widetilde{w}(x,t)\,dx=0,$$

for all $t\geq 0$. In the sequel, we shall work with $\widetilde{w}$ but we write w for simplicity.

The energy functional associated to (5), namely,

$$\mathscr{E}(t)=\frac{1}{2}\int_0^L\left\{p(x)u_x^2+m(x)u_t^2+c\theta^2+\tau w^2\right\}dx+\frac{|\mu|\,\tau_0}{2}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx, \tag{6}$$

we denote $\mathscr{E}(t)=\mathscr{E}(t,u,u_t,\theta,w,z)$ and $\mathscr{E}(0)=\mathscr{E}(0,u_0,u_1,\theta_0,w_0,f_0)$ for simplicity of notations. Then the energy E is decreasing function and satisfies, for and all $t\geq 0$,

$$\begin{aligned} \mathscr{E}'(t) &= -2\int_0^L\delta(x)u_{xt}^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx \\ &\quad -\mu\int_0^L u_tz_1(x,1,t)dx \\ &\leq (|\mu|-2\delta(\xi_6)/l)\int_0^L u_t^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx. \end{aligned}$$

To obtain precise decay rates of $E(t)$ as $t\to+\infty$, we assume that

$$m,\ \delta,\ p\in W^{1,\infty}(0,L),\ m(x),\ p(x),\ \delta(x)>0,\ \ \forall x\in[0,L]. \tag{7}$$

Let us introducing the vector function $U=(u,v,\theta,w,z)^T$, where $v=u_t$, using the standard Lebesgue space $L^2(0,L)$ and the Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms for define the spaces :

$$\mathscr{H}:=H_0^1(0,L)\times\left[L^2(0,L)\right]^2\times L_*^2(0,L)\times L^2((0,L)\times(0,1)),$$

and

$$H^2_*(0,L) = \left\{ w \in H^2(0,L) : w_x(L) = w_x(0) = 0 \right\},$$

where

$$L^2_*(0,L) = \left\{ w \in L^2(0,L) : \int_0^L w(s)\, ds = 0 \right\}.$$

We equip $\mathscr{H}$ with the inner product

$$\begin{aligned}(U,\widetilde{U})_{\mathscr{H}} &= \int_0^L p(x)\, u_x \widetilde{u}_x dx + \int_0^L m(x)\, v\widetilde{v} dx + c\int_0^L \theta\widetilde{\theta} dx + \tau \int_0^L w\widetilde{w} dx \\ &\quad + \tau_0 |\mu| \int_0^L \int_0^1 z\widetilde{z}(x,\rho,t) d\rho dx.\end{aligned}$$

Next, the system (5) can be reduced to the following abstract Cauchy problem :

$$\begin{cases} U'(t) + (\mathscr{A} + \mathscr{B})\, U(t) = 0, & t > 0 \\ U(0) = U_0 = (u_0, u_1, \theta_0, w_0, f_0)^T, \end{cases} \tag{8}$$

where the operator $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is defined by

$$\mathscr{A}U = \begin{pmatrix} -v \\ -\frac{1}{m(x)} \left[(p(x)u_x + 2\delta(x)v_x - \eta\theta - dw)_x - \mu z(x,1,t) - |\mu|\, v \right] \\ \frac{1}{c}\left(-k\theta_{xx} + \eta u_{tx} + k_1 w_x \right) \\ \frac{1}{\tau}\left(-k_3 w_{xx} + k_2 w + k_1 \theta_x + d u_{tx} \right) \end{pmatrix},$$

$$\mathscr{B}U = \begin{pmatrix} 0 \\ -\frac{|\mu| v}{m(x)} \\ 0 \\ 0 \end{pmatrix},$$

The domain of $\mathscr{A}$ is then

$$D(\mathscr{A}) = \left\{ \begin{array}{l} U \in \mathscr{H} \mid u \in H^2(0,L) \cap H^1_0(0,L), v \in H^1_0(0,L),\ \theta \in H^2(0,L), \\ w \in L^2_*(0,L) \cap H^2_*(0,L),\ z, z_\rho \in L^2((0,L) \times (0,1)), z(x,0) = u_t(x) \end{array} \right\},$$

which is dense in $\mathscr{H}$.

Proposition 1 *Let $U_0 \in \mathscr{H}$ be given. Problem (8) possesses then a unique solution satisfying $U \in C\left(\mathbb{R}^+; \mathscr{H}\right)$. If $U_0 \in D(\mathscr{A})$, then $U \in C^1\left(\mathbb{R}^+; \mathscr{H}\right) \cap C\left(\mathbb{R}^+; D(\mathscr{A})\right)$.*

Proof. Using semi-group theory. We prove that $\mathscr{A}$ is a maximal monotone operator and $\mathscr{B}$ is Lipschitz continuous in $\mathscr{H}$. Consequently, $\mathscr{A} + \mathscr{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathscr{H}$. ■

3. EXPONENTIAL STABILITY

In this section, we introduce some lemmas allow us to achieve our goal, which is the proof of the stability result.

Lemma 2 *[21] (Poincaré type Scheeffer's inequality) : Let $h \in H^1_0(0,L)$. Then it holds*

$$\int_0^L |h|^2\, dx \leq\ l \int_0^L |h_x|^2\, dx,\ \ l = \frac{L^2}{\pi^2}. \tag{9}$$

Lemma 3 *[2, 20] : Let (u, u_t, θ, w) be the solution to system (1)-(2), with an initial datum in $D(\mathscr{A})$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\xi_i \in [0, L] (i = 1, ..., 6)$, such that :*

$$\int_0^L p(x)\, u_x^2 dx = p(\xi_1) \int_0^L u_x^2 dx, \ \int_0^L \delta(x)\, u^2 dx = \delta(\xi_4) \int_0^L u^2 dx,$$

$$\int_0^L m(x)\, u_t^2 dx = m(\xi_2) \int_0^L u_t^2 dx, \ \int_0^L \delta(x)\, u_x^2 dx = \delta(\xi_5) \int_0^L u_x^2 dx,$$

$$\int_0^L m(x)\, u^2 dx = m(\xi_3) \int_0^L u^2 dx, \ \int_0^L \delta(x)\, u_{xt}^2 dx = \delta(\xi_6) \int_0^L u_{xt}^2 dx.$$

Lemma 4 *Let (u, v, θ, w, z) be the solution of (5), then the energy $\mathscr{E}$ is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned} \mathscr{E}'(t) &= -2\int_0^L \delta(x)\, u_{xt}^2 dx - k_2 \int_0^L w^2 dx - k_3 \int_0^L w_x^2 dx - k \int_0^L \theta_x^2 dx - \mu \int_0^L u_t z_1(x,1,t) dx \\ &\leq (|\mu| - 2\delta(\xi_6)/l) \int_0^L u_t^2 dx - k_2 \int_0^L w^2 dx - k_3 \int_0^L w_x^2 dx - k \int_0^L \theta_x^2 dx. \end{aligned} \quad (10)$$

Proof. Multiplying the equations in $(5)_1$,$(5)_2$, $(5)_3$ and $(5)_4$ by u_t, θ, w and $|\mu| z$, respectively, integrate over $(0, L)$ and $(0, L) \times (0, 1)$ and using (9), we obtain (10). ■

Lemma 5 *The functional*

$$I_1(t) := \int_0^L \left(\delta(x) u_x^2 + m(x) u_t u \right) dx, \quad (11)$$

satisfies

$$\begin{aligned} I_1'(t) &\leq -\left(p(\xi_1) - \mu^2 l - (\eta + d)\, \varepsilon_1 \right) \int_0^L u_x^2 dx + m(\xi_2) \int_0^L u_t^2 dx + \frac{\eta}{4\varepsilon_1} \int_0^L \theta^2 dx \\ &\quad + \frac{d}{4\varepsilon_1} \int_0^L w^2 dx + \frac{1}{4} \int_0^L z^2(x,1,t) dx, \end{aligned} \quad (12)$$

for any $\varepsilon_1 > 0$.

Proof. Differentiating Equation (11) with respect to t and using Equations $(1)_1$. By using Young's inequality (see [4] p. 92) and Lemma 3, we complete the proof. ■

Lemma 6 *The functional*

$$I_2(t) := \tau c \int_0^L \theta \left(\int_0^x w(y) dy \right) dx, \quad (13)$$

satisfies

$$\begin{aligned} I_2'(t) &\leq (-k_1 c + 3\varepsilon_2) \int_0^L \theta^2 dx + \frac{1}{2\varepsilon_2} \int_0^L u_t^2 dx + \frac{1}{4\varepsilon_2} \int_0^L \theta_x^2 dx \\ &\quad + \left(k_1 \tau + 2\varepsilon_2 c' + c' \right) \int_0^L w^2 dx + \frac{1}{4\varepsilon_2} \int_0^L w_x^2 dx, \end{aligned} \quad (14)$$

for any $\varepsilon_2 > 0$.

Proof. Taking the derivative of (13) and using $(1)_2$ and $(1)_3$, Integration by parts and using Young's inequality, we infer (14). ■

Lemma 7 *The functional*

$$I_3(t) = \tau_0 \int_0^L \int_0^1 e^{-2\tau_0 \rho} z^2(x,\rho,t) d\rho dx \tag{15}$$

satisfies

$$I_3'(t) \leq -2I_3(t) - c' \int_0^L z_1^2(x,1,t)dx + \int_0^L \varphi_t^2 dx \tag{16}$$

for $c' > 0$.

Proof. *By differentiating* I_3*, then using* $(5)_5$ *and integrating by parts.* ■

Next, we define a Lyapunov functional $\mathscr{L}$ and show that it is equivalent to the energy functional.

Lemma 8 *For N sufficiently large, the functional defined by*

$$\mathscr{L}(t) := N\mathscr{E}(t) + I_1(t) + N_1 I_2(t) + N_2 I_3(t). \tag{17}$$

where N and N_1 *are positive real numbers to be chosen appropriately later, satisfies*

$$c_1' \mathscr{E}(t) \leq \mathscr{L}(t) \leq c_2' \mathscr{E}(t), \quad \forall t \geq 0. \tag{18}$$

where c_1' *and* c_2' *are positive constants.*

Now, we are ready to state and prove the main result of this section.

Theorem 9 *Let* (u,v,θ,w,z) *be the solution of (5), then the energy* $\mathscr{E}$ *satisfies, for all* $t \geq 0$,

$$\mathscr{E}(t) \leq c_1 e^{-c_2 t},$$

where c_1 *and* c_2 *are positive constants.*

Proof. We differentiate (17), and recall (10), (12), and (14), we obtain

$$\begin{aligned}
\mathscr{L}'(t) \quad \leq \quad & \left\{ N\left(|\mu| - 2\delta(\xi_6)/l\right) + \frac{N_1}{2\varepsilon_2} + m(\xi_2) + N_2 \right\} \int_0^L u_t^2 dx \\
& + \left\{ \left(\mu^2 l - p(\xi_1)\right) + (\eta + d)\varepsilon_1 \right\} \int_0^L u_x^2 dx \\
& + \left\{ -Nk_2 + N_1\left(k_1 \tau + 2\varepsilon_2 c' + c'\right) + \frac{d}{4\varepsilon_1} \right\} \int_0^L w^2 dx \\
& + \left\{ N_1\left(-k_1 c + 3\varepsilon_2\right) + \frac{\eta}{4\varepsilon_1} \right\} \int_0^L \theta^2 dx + \left\{ -Nk + \frac{N_1}{4\varepsilon_2} \right\} \int_0^L \theta_x^2 dx \\
& + \left\{ -Nk_3 + \frac{N_1}{4\varepsilon_2} \right\} \int_0^L w_x^2 dx + \left\{ \frac{1}{4} - c' N_2 \right\} \int_0^L z^2(x,1,t) dx
\end{aligned}$$

At this point, we choose ε_2 small enough such that

$$-k_1 c + 3\varepsilon_2 < 0,$$

then, we choose $|\mu|$ small enough such that

$$|\mu| - 2\delta(\xi_6)/l < 0, \; \mu^2 l - p(\xi_1) < 0$$

Now, we take ε_1 small enough such that

$$-p(\xi_1)+\mu^2 l+(\eta+d)\,\varepsilon_1<0$$

then, choosing N_1 large enough so that

$$N_1\,(-k_1c+3\varepsilon_2)+\frac{\eta}{4\varepsilon_1}<0.$$

Once N_1 is fixed, we then choose N_2 large enough so that

$$\frac{1}{4}-c'N_2<0.$$

Finally, we then choose N large enough so that

$$\begin{array}{l} -Nc'+\frac{N_1}{2\varepsilon_2}+m(\xi_2)+N_2<0, \\ -Nk_2+N_1\,(k_1\tau+2\varepsilon_2c'+c')+\frac{d}{4\varepsilon_1}<0, \\ -Nk+\frac{N_1}{4\varepsilon_2}<0,\ -Nk_3+\frac{N_1}{4\varepsilon_2}<0. \end{array}$$

Thus, using (6), we arrive at

$$\mathscr{L}'(t)\leq -c\mathscr{E}(t),\quad \forall t>0. \tag{19}$$

A combination of (18) and (19) gives

$$\mathscr{L}'(t)\leq -c_2\mathscr{L}(t),\quad \forall t>0. \tag{20}$$

where $c_2=c/c_2'$, a simple integration of (20) over $(0,t)$ yields

$$c_1'\mathscr{E}(t)\leq \mathscr{L}(t)\leq \mathscr{L}(0)e^{-c_2t}\ ,\ \forall t>0.$$

Taking $c_1=\mathscr{L}(0)/c_1'$ which completes the proof. ■

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