A NOTE ON THE INFLUENCE OF DIFFERENT ADDITIONAL REGULARITY ON THE CRITICAL EXPONENT

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ABSTRACT

In this paper, we consider the Cauchy problem for the semi-linear damped σ-evolution equations, where the initial data are supposed to belong to the energy space with different additional regularity, which means that,

$$(u_0,u_1)\in\Big(H^{\sigma}(\mathbb{R}^n)\cap L^{m_1}(\mathbb{R}^n)\Big)\times\Big(L^2(\mathbb{R}^n)\cap L^{m_2}(\mathbb{R}^n)\Big),\ \ m_1,m_2\in[1,2),\ \sigma\geq 1.$$

Our main goal is to study the influence of m_1 and m_2 on the critical exponent by proving the global (in time) existence of small data energy solutions where their decay estimates coincide with those to the corresponding linear equation.

1. INTRODUCTION

The semi-linear damped σ-evolution equations we want to study in this paper are :

$$\partial_t^2 u+(-\Delta)^{\sigma}u+\partial_t u+(-\Delta)^{\sigma}\partial_t u=|u|^p,\ \ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x), \tag{1}$$

with $\sigma\in[1,\infty)$, $p\in(1,\infty)$, $t\in[0,\infty)$ and $x\in\mathbb{R}^n$, $n\geq 1$.
Here, the notation $(-\Delta)^{\sigma}$ denotes the fractional Laplacian operator with symbol $|\xi|^{2\sigma}$, i.e.,

$$\mathscr{F}\left((-\Delta)^{\sigma}f\right)=|\xi|^{2\sigma}\mathscr{F}(f)(\xi),\ \ \xi\in\mathbb{R}^n,\ \ |\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2},$$

where $\mathscr{F}$ is the Fourier transform. The terms $\partial_t(\cdot)$ and $(-\Delta)^{\sigma}\partial_t(\cdot)$ they respectively denote frictional and visco-elastic damping mechanism. In this paper we will choose the initial data (u_0,u_1) that belong to the energy space $H^{\sigma}(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ with different additional regularity, that is,

$$u_0\in H^{\sigma}(\mathbb{R}^n)\cap L^{m_1}(\mathbb{R}^n),\ \ u_1\in L^2(\mathbb{R}^n)\cap L^{m_2}(\mathbb{R}^n),\ \ m_1,m_2\in[1,2). \tag{2}$$

Our main goal is to study the influence of m_1 and m_2 not only on the critical exponent but on the decay estimates of solutions u as well.

Indeed, critical exponent p_{crit} means global (in time) existence of small data Sobolev solutions for $p>p_{crit}$, and blow-up in finite time for $1<p\leq p_{crit}$. Additionally, when $p>p_{crit}$ the decay estimates for solution of the semi-linear Cauchy problem are the same for those of the linear problem.

The pioneering paper [1] is the first to deal with the problem of finding the critical exponent p_{crit} where the initial data (v_0,v_1) are small in $\big(H^1(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)\times\big(L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\big)$ and $m\in[1,2]$, the authors in [1] studied the existence property of solutions to the Cauchy problem for the semi-linear wave equation with frictional damping

$$\partial_t^2 v-\Delta v+\partial_t v=|u|^p,\ \ v(0,x)=v_0(x),\ \partial_t v(0,x)=v_1(x),$$

and they found the number $p_{crit}(n,m)$ which divides the range of p into $p \in (1, p_{crit}(n,m))$ (the small data global nonexistence) and $p \in (p_{crit}(n,m), \infty)$ (the small data global existence), where

$$p_{crit}(n,m) = 1 + \frac{2m}{n},\ m \in [1,2]. \tag{3}$$

It is clear that $p_{crit}(n,m)$ interpolates the critical exponents $p_{crit}(n,1)$ and $p_{crit}(n,2)$. Here, we note that the number $p_{crit}(n,1)$ is well-known as Fujita exponent which was first found by Hiroshi Fujita for the semilinear heat equation $h_t - \Delta h = h^p$, $h(0,x) = h_0(x)$, $p > 1$.

More recently, the authors in [2] used unified $(L^m \cap L^2) - L^2$ linear estimates to prove the global (in time) existence of small data solutions for the problem (1) and they found the following critical exponent :

$$p_{crit}(n,m,\sigma) = 1 + \frac{2m\sigma}{n},\ \ m \in [1,2), \tag{4}$$

where they choose the initial data (u_0, u_1) as in (2) with $m_1 = m_2 = m$.
Since $p_{crit}(n,m,\sigma)$ is always depends on the parameter m, this fact leads us to ask the following question :

If we choose the initial data as in (2), what happens to the critical exponent (4) ?

To answer this question, we will prove in this paper the global (in time) existence of small data solutions to (1) where our method is standard and is based on Banach fixed point theorem, Gagliardo-Nirenberg inequality as well as the application of mixed $L^m - L^2$ linear estimates.

For the best reading of this paper, we have the following notation :

— We write $f \lesssim g$ when there exists a constant $c > 0$ such that $f \leq cg$.

— $H^\sigma(\mathbb{R}^n)$ mean Sobolev spaces as defined below :

$$H^\sigma(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{H^\sigma(\mathbb{R}^n)} = \|(1+|\cdot|^2)^{\frac{\sigma}{2}} \mathscr{F}(f)\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

— We denote $L^m(\mathbb{R}^n)$ the usual Lebesgue spaces, with $m \in [1,2)$.

2. LINEAR ESTIMATES

The so-called $L^m - L^2$ linear estimates is a very important tool to demonstrate Theorem 3. We recall them in the following proposition.

Lemma 1 (Proposition 2.1 [2]) *Let $m \in [1,2)$. Then, the Sobolev solutions u^{lin} to the following linear equation :*

$$\partial_t^2 u^{lin} + (-\Delta)^\sigma u^{lin} + \partial_t u^{lin} + (-\Delta)^\sigma \partial_t u^{lin} = 0,\ u^{lin}(0,x) = u_0(x),\ \partial_t u^{lin}(0,x) = u_1(x), \tag{5}$$

satisfy the $(L^m \cap L^2) - L^2$ estimates :

$$\begin{aligned} \|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \lesssim\ & (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_0\|_{(L^m \cap H^a)} \\ & + (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_1\|_{(L^m \cap H^{[a+2(j-1)\sigma]^+})}, \end{aligned} \tag{6}$$

and the $L^2 - L^2$ estimates :

$$\|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{H^a \times H^{[a+2(j-1)\sigma]^+}}, \tag{7}$$

for any $a \geq 0$, $j = 0,1$ and for all space dimensions $n \geq 1$, where $[\cdot]^+ = \max\{0,\cdot\}$.

Now, recalling the data spaces (2), we may obtain the following corollary.

Corollary 2 *Let $m_1, m_2 \in [1,2)$. Then, the Sobolev solutions u^{lin} to the linear equation (5) satisfy the $(L^m \cap L^2) - L^2$ estimates :*

$$\|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_0\|_{(L^{m_1}\cap H^a)} + (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_1\|_{(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})}$$

$$\lesssim \begin{cases} (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{(L^{m_1}\cap H^a)\times(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})} & if \;\; m_2 \le m_1, \\ (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{(L^{m_1}\cap H^a)\times(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})} & if \;\; m_1 \le m_2, \end{cases} \tag{8}$$

as well as (7).

3. MAIN RESULTS

Our main results are divided into two cases,

$$m_1 \le m_2 \;\text{ and } m_2 \le m_1.$$

In the following theorem we will see the nice influence of m_1 and m_2 on the critical exponent (4) when $m_2 \le m_1$.

Theorem 3 *Let us consider the Cauchy problem (1) with $\sigma \ge 1$ and $p > 1$. Let $m_1, m_2 \in [1,2)$ such that*

$$m_2 \le m_1.$$

We assume the following conditions for p and the dimension n :

$$\begin{cases} \frac{2}{m_2} \le p \le \frac{n}{n-2\sigma} & if \;\; 2\sigma < n \le \frac{4\sigma}{2-m_2}, \\ \frac{2}{m_2} \le p & if \;\; 1 \le n \le 2\sigma. \end{cases} \tag{9}$$

Moreover, we suppose

$$p > \frac{m_1}{m_2} + \frac{2m_1\sigma}{n}. \tag{10}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any data

$$(u_0,u_1) \in \mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n) := (H^\sigma(\mathbb{R}^n)\cap L^{m_1}(\mathbb{R}^n)) \times \left(L^2(\mathbb{R}^n)\cap L^{m_2}(\mathbb{R}^n)\right),$$

with $\|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}} < \varepsilon_0$, we have a uniquely determined globally (in time) solution

$$u \in \mathscr{C}\left([0,\infty), H^\sigma(\mathbb{R}^n)\right) \cap \mathscr{C}^1\left([0,\infty), L^2(\mathbb{R}^n)\right)$$

to (1). Furthermore, the solution satisfies the estimates :

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\left\|(-\Delta)^{\sigma/2} u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)}.$$

In the following theorem, when $m_1 \leq m_2$, then the critical exponent (4) can only be influenced by the parameter m_2 of the additional regularity of the second initial data (initial velocity).

Theorem 4 *Let us consider the Cauchy problem (1) with $\sigma \geq 1$ and $p > 1$. Let $m_1, m_2 \in [1,2)$ such that*

$$m_1 \leq m_2.$$

We assume the same conditions for p and the dimension n as in (9). Moreover, we suppose

$$p > 1 + \frac{2m_2\sigma}{n}. \tag{11}$$

Then we have the same conclusion as in Theorem (3). Furthermore, the solution u satisfies the estimates :

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\left\|(-\Delta)^{\sigma/2}u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}} \|(u_0,u_1)\|_{\mathscr{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)}.$$

Remark 1 *The conditions (10), (11) are assumed to get the same decay estimates of the semi-linear model with those of the corresponding linear model (5). The bounds (9) on p and n appear due to the application of Gagliardo-Nirenberg inequality.*

4. CONCLUSIONS

We have proven in this paper how the different additional L^m regularity of the initial data could possibly affect the critical exponent and also the decay estimates of the solutions to the semi-linear Cauchy problem (1). In other words, Theorem (3) showed the influence of the additional L^{m_1} regularity of u_0 not only on the critical exponent but also on the decay estimates. While, the decay estimates in the second theorem are related to u_1 (see again Corollary 2). It is clear that when $m_1 = m_2$ our results in Theorems 3, 4 coincide with those in the cited paper [2].

Because the two exponents of the global existence (10) and (11) are depend by the two parameters of additional regularity m_1 and m_2, the author ask the following question about a blow-up result :

Question :

Does the solution u blows-up if the exponent p satisfies

$$p < 1 + \frac{2m_2\sigma}{n} \;\; \textit{if} \;\; m_1 \leq m_2, \;\; \textit{or} \;\; p < \frac{m_1}{m_2} + \frac{2m_1\sigma}{n} \;\; \textit{if} \;\; m_2 \leq m_1?$$

5. REFERENCES

[1] R. Ikehata and M. Ohta, Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^n$, J. Math. Anal. Appl., 269(2002), 87-97.

[2] T.A. Dao and H. Michihisa, Study of semi-linear σ-evolution equations with frictional and visco-elastic damping, Comm. Pure. Appl. Anal., 19(2020), 1581–1608.